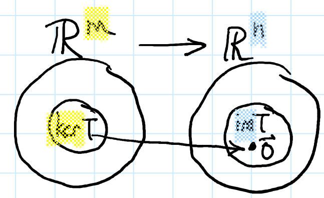
12-Mar L12

Wednesday, March 12, 2025 1:31 PM



tying concepts together:

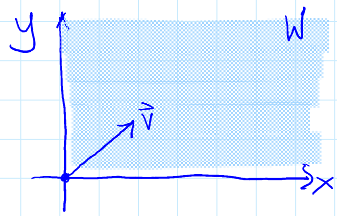
given $T(\vec{x}) = A\vec{x}$ is LT $\mathbb{R}^m \to \mathbb{R}^n$

then 1. $\ker(T) = \ker(A)$ is a subspace of $\mathbb{R}^m$

2. $\mathrm{im}(T) = \mathrm{im}(A)$ " of $\mathbb{R}^n$

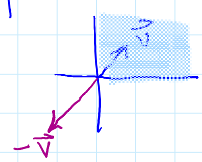
ex. Is $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \text{ s.t. } x \geq 0,\ y \geq 0 \right\}$ a subspace of $\mathbb{R}^2$?

contains $\vec{0}$



closed under addition

however if pick scalar of -1 then



W is not closed under scalar mult.

subspaces of $\mathbb{R}^2$: $\{\vec{0}\}$ $\to$ 0 $\mathbb{R}^1$ / $\to (0,0)$ in $\mathbb{R}^2$ or $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

lines through $\vec{0}$

$\mathbb{R}^2$ itself

subspaces of $\mathbb{R}^3$: $\{\vec{0}\}$

lines through $\vec{0}$

planes through $\vec{0}$

$\mathbb{R}^3$ itself

ex given plane $V \in \mathbb{R}^3$ s.t. $x_1 + 2x_2 + 3x_3 = 0$

- find a matrix A s.t. $V = \ker(A)$

can be written as

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

format of kernel $\longrightarrow$ $A \vec{x} = 0$

$\therefore V = \ker(A)$
$= \ker \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$

- find a matrix B st. $V = \mathrm{im}(B)$

since image of a matrix is a span of its columns, V can be written as a span of some vectors —

since image of a matrix is a span of its columns,
V can be written as a span of some vectors —
specifically can choose any two non-parallel vectors

choose $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \Rightarrow -2 + 2(1) + 3(0) = 0$

choose $\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \Rightarrow -3 + 2(0) + 3(1) = 0$

$\longrightarrow V = \text{im}(B) = \begin{bmatrix} -2 & -3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$

note: sometimes a subspace that was defined as a kernel must be expressed as an image or vice versa use G-J elim.

given $\underset{3\times4}{A} = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 2 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$ find vectors in $\mathbb{R}^3$ that span $\text{im}(A)$

recall: $\text{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4) = \{ c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + c_4\vec{v}_4 \}$
i.e., set all of LC

then $\text{im}(A)$ is spanned by $\underset{\vec{v}_1}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}, \underset{\vec{v}_2}{\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}}, \underset{\vec{v}_3}{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}, \underset{\vec{v}_4}{\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}}$ ← vectors in $\mathbb{R}^3$ bc hv. 3 components

however, don't need all 4 vectors to express span since

$\vec{v}_2 = 2\vec{v}_1$ and $\vec{v}_4 = \vec{v}_1 + \vec{v}_3$ $\quad \vec{v}_2, \vec{v}_4$ are redundant vectors

$\therefore \text{im}(A) = \text{span}(\vec{v}_1, \vec{v}_3)$

verify that $\text{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4) = \text{span}(\vec{v}_1, \vec{v}_3)$

$$\begin{aligned}
\vec{v} &= c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + c_4\vec{v}_4 \\
&= c_1\vec{v}_1 + c_2 \cdot 2\vec{v}_1 + c_3\vec{v}_3 + c_4(\vec{v}_1 + \vec{v}_3) \\
&= \underline{c_1\vec{v}_1} + \underline{2c_2\vec{v}_1} + c_3\vec{v}_3 + \underline{c_4\vec{v}_1} + c_4\vec{v}_3 \\
&= (c_1 + 2c_2 + c_4)\vec{v}_1 + (c_3 + c_4)\vec{v}_3
\end{aligned}$$
$\square$

conclude that $\vec{v}_1, \dots, \vec{v}_m$ in $\mathbb{R}^n$ are linearly independent if none of the vectors are redundant and none of the vectors are redundant and are linearly dependent if at least one vector is redundant

furthermore, $\vec{v}_1, \dots, \vec{v}_m$ in a subspace of V in $\mathbb{R}^n$ form a basis of V if
- spans V
- are linearly independent

ex. in previous example, $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\vec{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ are linearly independent

$\therefore$ im(A) = basis of V

thm: construct a basis of im(A) by listing all non-redundant vectors of A

Q: How to list this set of vectors?

Method #1: by inspection

ex. are following vectors in $\mathbb{R}^7$ linearly independent?

$$\begin{bmatrix} 7 \\ 0 \\ 4 \\ 0 \\ 1 \\ 9 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ 0 \\ 7 \\ 1 \\ 4 \\ 8 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ 6 \\ 2 \\ 3 \\ 1 \\ 7 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 3 \\ 3 \\ 2 \\ 2 \\ 4 \end{bmatrix}$$

technique:
check rows with components of zero, entries to the left of 1st non-zero component should all be zero

if this is the case for all rows, then vectors are linearly independent

(not always applicable ...)

Method 2: use G-J elim.

ex. are following vectors linearly independent?

$$\overset{\vec{v}_1}{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}, \overset{\vec{v}_2}{\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}}, \overset{\vec{v}_3}{\begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}}$$

Method 1 is irrelevant since $\nexists$ no zero entries / components

$\vec{v}_1$ and $\vec{v}_2$ are not redundant bc $\cdot$ $\vec{v}_1$ is non-zero and

$\vec{v}_1$ and $\vec{v}_2$ are <u>not</u> redundant bc • $\vec{v}_1$ is non-zero and $\vec{v}_2$ is not scalar of $\vec{v}_1$

check if $\vec{v}_3$ is LC of $\vec{v}_1$ and $\vec{v}_2$

use an augmented matrix:

$$M = \left[\begin{array}{cc|c} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{array}\right] \begin{array}{l} \text{I} \\ \text{II} \\ \text{III} \end{array}$$

$$c_1\vec{v}_1 + c_2\vec{v}_2 \overset{?}{=} \vec{v}_3$$

$$\downarrow$$

$$\begin{array}{l} \text{I} \\ \text{IV} \\ \end{array} \left[\begin{array}{cc|c} 1 & 4 & 7 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$$

$$\begin{array}{r|ccc} -2\text{I} & -2 & -8 & -14 \\ \text{II} & 2 & 5 & 8 \\ \hline & 0 & -3 & -6 \end{array} \rightarrow 0 \quad 1 \quad 2 \quad \text{IV}$$

$$\begin{array}{r|ccc} -3\text{I} & -3 & -12 & -21 \\ \text{III} & 3 & 6 & 9 \\ \hline & 0 & -6 & -12 \end{array} \rightarrow 0 \quad 1 \quad 2$$

$$\begin{array}{r|ccc} \text{I} & 1 & 4 & 7 \\ -4\text{IV} & 0 & -4 & -8 \\ \hline & 1 & 0 & -1 \end{array}$$

$$\text{rref}(M) = \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right] \leftarrow \text{consistent}$$

$c_1 = -1 \quad c_2 = 2 \quad \Rightarrow \vec{v}_3 = -1\vec{v}_1 + 2\vec{v}_2$

<u>can</u> write $\vec{v}_3$ as LC of $\vec{v}_1$, $\vec{v}_2$

conclude: $\vec{v}_3$ redundant

next, write equation as <u>linear relation</u> by solving for $\vec{0}$

hence, $\vec{v}_3 = -\vec{v}_1 + 2\vec{v}_2$ becomes $\vec{v}_1 - 2\vec{v}_2 + \vec{v}_3 = \vec{0}$

always exists a <u>trivial</u> relation where $c_1 = \dots = c_m = 0$

then <u>non-trivial</u> relation is one that contains at least one non-zero coefficient, c_i

finally, $\vec{v}_1, \dots, \vec{v}_m$ in $\mathbb{R}^n$ are linearly dependent iff $\exists$ non-trivial relations among them

ex. given that $n \times m$ matrix A contains column vectors that are linearly <u>in</u>dependent then $A\vec{x} = \vec{0}$ can be written as

$$\begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = 0$$

$\to x_1\vec{v}_1 + \ldots + x_m\vec{v}_m = 0$

thm: if column vectors are linearly independent then $x_1 = \ldots = x_m = 0$ and $\ker(A) = \{\vec{0}\}$

see above re. linear dependence

Likewise, column vectors of A are linearly independent iff $\ker(A) = \{\vec{0}\}$ or $\operatorname{rank}(A) = m$

$\downarrow$ implies

$m \leq n$

conclude: exists at most n linearly independent vectors in $\mathbb{R}^n$

revisit $A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$ found that $\begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} = -1\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 2\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$

redundant

relation among column vectors:

$$1\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - 2\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} + 1\begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} = \vec{0}$$

$\therefore$ can conclude that $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ is in $\ker(A)$ since $A\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

revisit def'n of basis:

given $\vec{v}_1, \ldots, \vec{v}_m$ in a subspace of V in $\mathbb{R}^n$, these vectors form a basis of V iff every vector $\vec{v} \in V$ can be expressed uniquely as LC $\vec{v} = c_1\vec{v}_1 + \ldots + c_m\vec{v}_m$

later, basis coefficients $c_1, \ldots, c_m$ will be called coordinates